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Some inequalities concerning geometric constants of Banach spaces

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Let X be a real Banach space with $\dim X \geq 2$. The closed unit ball and unit sphere of X are denoted by B_X and S_X , respectively. We shall consider the following constants:

$$C_{NJ}(X) = \sup \left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X \right\}, \quad (1)$$

$$C'_{NJ}(X) = \sup \left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{4} : x, y \in S_X \right\}, \quad (2)$$

$$C_Z(X) = \sup \left\{ \frac{\|x+y\|\|x-y\|}{\|x\|^2 + \|y\|^2} : x \in S_X, y \in B_X \right\}, \quad (3)$$

$$C'_Z(X) = \sup \left\{ \frac{\|x+y\|\|x-y\|}{2} : x, y \in S_X \right\}, \quad (4)$$

$$J(X) = \sup \{ \min(\|x+y\|, \|x-y\|) : x, y \in S_X \}, \quad (5)$$

$$\rho_X(1) = \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : x, y \in S_X \right\}. \quad (6)$$

Properties and relations concerning these constants have been studied by many authors. In particular the *James constant* $J(X)$ and the *von Neumann-Jordan constant* $C_{NJ}(X)$ have been most widely treated. Recall that a Banach space X is *uniformly non-square* provided $J(X) < 2$ or equivalently $C_{NJ}(X) < 2$. The constant $C'_{NJ}(X)$ may be considered as the unitary version of $C_{NJ}(X)$ ([2],[6]). The constant $C_Z(X)$ was introduced by Zbăganu [11], who conjectured that $C_{NJ}(X) = C_Z(X)$ for all Banach spaces X , but in general these two constants are different.

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The constant $C'_Z(X)$ may be considered as the unitary version of $C_Z(X)$ ([6]). Then we easily have

$$\frac{J(X)^2}{2} \leq C'_Z(X) \leq C_Z(X) \leq C_{NJ}(X), \quad (7)$$

$$\frac{J(X)^2}{2} \leq C'_Z(X) \leq \frac{(1 + \rho_X(1))^2}{2} \leq C'_{NJ}(X) \leq C_{NJ}(X). \quad (8)$$

There are many Banach spaces X for which all the terms in (7) and (8) coincide. On the other hand, there exists a Banach space X such that $C_Z(X) < (1 + \rho_X(1))^2/2$ and $C_Z(X^*) > C'_{NJ}(X^*)$, where X^* is the dual space of X ([6, Example 3]).

We shall start with the estimate of $C_{NJ}(X)$ by $C'_{NJ}(X)$ and $\rho_X(1)$, which was recently proved in Kato and Takahashi [5]. In the following, let $f(u, v) = u - v + \sqrt{(u - v - 1)^2 + v^2}$. Then it is easy to see that for all $u_2 \geq u_1 \geq 1$ and $v_2 \geq v_1 \geq u_1 - 1$,

$$f(u_1, v_1) \leq f(u_2, v_2),$$

where equality holds only when $u_1 = u_2$ and $v_1 = v_2$ ([5, Lemma 3.6]).

Theorem 1 ([5, Theorem 3.4]). *Let X be a Banach space. Then*

$$C_{NJ}(X) \leq f(C'_{NJ}(X), \rho_X(1)) \quad (9)$$

or

$$C_{NJ}(X) \leq C'_{NJ}(X) - \rho_X(1) + \sqrt{(C'_{NJ}(X) - \rho_X(1) - 1)^2 + \rho_X(1)^2}.$$

If X is not uniformly non-square, we have equality in (9) as $C_{NJ}(X) = C'_{NJ}(X) = 2$ and $\rho_X(1) = 1$. The inequality (9) also attains equality with both of the Day-James ℓ_∞ - ℓ_1 and ℓ_2 - ℓ_∞ spaces, which are uniformly non-square ([5, Remark 3.5]). Let us mention that Theorem 1 yields some previous results concerning the estimates of $C_{NJ}(X)$ by $C'_{NJ}(X)$ and by $\rho_X(1)$. To see this we need the following inequalities

$$\frac{(1 + \rho_X(1))^2}{2} \leq C'_{NJ}(X) \leq 1 + \rho_X(1)^2, \quad (10)$$

see [5, Proposition 3.2]. Since $\rho_X(1) \leq \sqrt{2C'_{NJ}(X)} - 1$, by (9) we have

Corollary 1 ([2, Theorem 1]). *Let X be a Banach space. Then*

$$C_{NJ}(X) \leq f(C'_{NJ}(X), \sqrt{2C'_{NJ}(X)} - 1),$$

which is written as

$$C_{NJ}(X) \leq 1 + \left(\sqrt{2C'_{NJ}(X)} - 1 \right)^2. \quad (11)$$

Since $C'_{NJ}(X) \leq 1 + \rho_X(1)^2$, by (9) we also have

Corollary 2 ([7, Theorem 2]). *Let X be a Banach space. Then*

$$C_{NJ}(X) \leq f(1 + \rho_X(1)^2, \rho_X(1)),$$

which is written as

$$C_{NJ}(X) \leq 1 + \rho_X(1) \left(\sqrt{(1 - \rho_X(1))^2 + 1} - (1 - \rho_X(1)) \right). \quad (12)$$

We shall present the estimate of $C_{NJ}(X)$ by $C'_Z(X)$. To do this we need the estimates of $\rho_X(1)$ and $C'_{NJ}(X)$ by $C'_Z(X)$.

Proposition 1. *For any Banach space X ,*

$$\rho_X(1) \leq \frac{C'_Z(X)}{2}, \quad (13)$$

$$C'_{NJ}(X) \leq 1 + \frac{C'_Z(X)^2}{4}. \quad (14)$$

In (13) and (14) equalities are attained if X is not uniformly non-square. If X is the ℓ_2 - ℓ_1 space, we also have equalities in (13) and (14) as $\rho_X(1) = 1/\sqrt{2}$, $C'_{NJ}(X) = 3/2$ and $C'_Z(X) = \sqrt{2}$ ([6, Example 3]).

By (9), (13) and (14) we have

Theorem 2. *Let X be a Banach space. Then*

$$C_{NJ}(X) \leq f(1 + C'_Z(X)^2/4, C'_Z(X)/2),$$

which is written as

$$C_{NJ}(X) \leq \frac{C'_Z(X)^2}{4} + 1 + \frac{C'_Z(X)}{4} \left(\sqrt{C'_Z(X)^2 - 4C'_Z(X) + 8} - 2 \right). \quad (15)$$

Now we shall present the estimate of $C_Z(X)$ by $C'_Z(X)$ and $\rho_X(1)$, which is similar to Theorem 1.

Theorem 3. *Let X be a Banach space. Then*

$$C_Z(X) \leq f(C'_Z(X), \rho_X(1)) \quad (16)$$

or

$$C_Z(X) \leq C'_Z(X) - \rho_X(1) + \sqrt{(C'_Z(X) - \rho_X(1) - 1)^2 + \rho_X(1)^2}.$$

Since $C'_Z(X) \leq (1 + \rho_X(1))^2/2$, by (16) we have

Corollary 3. *For any Banach space X*

$$C_Z(X) \leq f((1 + \rho_X(1))^2/2, \rho_X(1)) = 1 + \rho_X(1)^2. \quad (17)$$

Remark 1. The estimates given by Theorem 3 and Corollary 3 are sharp. It should be noted that if $C_Z(X) = 1 + \rho_X(1)^2$, we have equality in (16). It is clear that if X is not uniformly non-square, we have equality in (17) as $C_Z(X) = 2$ and $\rho_X(1) = 1$. The inequality (17) also attains equality with both of the Day-James ℓ_∞ - ℓ_1 and ℓ_2 - ℓ_∞ spaces, which are uniformly non-square. In fact, $C_Z(X) = 5/4$, $\rho_X(1) = 1/2$ if X is the ℓ_∞ - ℓ_1 space ([6, Example 4]), and $C_Z(X) = 3/2$, $\rho_X(1) = 1/\sqrt{2}$ if X is the ℓ_2 - ℓ_∞ space ([6, Example 3]).

By (13) and (16) we have

Theorem 4. *Let X be a Banach space. Then*

$$C_Z(X) \leq f(C'_Z(X), C'_Z(X)/2),$$

which is written as

$$C_Z(X) \leq \frac{1}{2} \left\{ C'_Z(X) + \sqrt{(C'_Z(X) - 2)^2 + C'_Z(X)^2} \right\}. \quad (18)$$

Finally we shall present the estimates of $C'_Z(X)$ and $C_Z(X)$ by $J(X)$.

Theorem 5. *Let X be a Banach space. Then*

$$C'_Z(X) \leq 4(1 - 1/J(X)). \quad (19)$$

The estimate (19) yields that $\rho_X(1) \leq C'_Z(X)/2 \leq 2(1 - 1/J(X))$ ([7, Theorem 1]).

By Theorems 2 and 5 we shall have the quite simple inequality $C_{NJ}(X) \leq J(X)$, which was proved in 2009 by Takahashi and Kato [7] (see also [5, 8, 10]).

Corollary 4. *Let X be a Banach space. Then*

$$C_{NJ}(X) \leq \frac{4}{4 - C'_Z(X)} \leq J(X). \quad (20)$$

By (14), (15), (18) and (19) we have

Theorem 6. *Let X be a Banach space and let $J = J(X)$. Then*

$$C_{NJ}(X) \leq \left(\frac{\sqrt{2}(J-1) + \sqrt{(J-1)^2 + 1}}{J} \right)^2, \quad (21)$$

$$C'_{NJ}(X) \leq 1 + 4 \left(1 - \frac{1}{J} \right)^2, \quad (22)$$

$$C_Z(X) \leq \frac{2(J-1) + \sqrt{5J^2 - 12J + 8}}{J}. \quad (23)$$

Remark 2. Let $J = J(X) < 2$. Then it is easy to see that

$$\frac{2(J-1) + \sqrt{5J^2 - 12J + 8}}{J} < 1 + 4(1 - 1/J)^2 < \left(\frac{\sqrt{2}(J-1) + \sqrt{(J-1)^2 + 1}}{J} \right)^2.$$

The estimates (21) and (22) was proved by Kato and Takahashi [5], see also Wang [8]. On the other hand, the estimate (23) is new.

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